Adiabatic motion of two-component BPS kinks

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Abstract

The low energy dynamics of degenerated BPS domain walls arising in a generalized Wess-Zumino model is described as geodesic motion in the moduli space of these topological walls

1 Introduction

According to a fruitful idea by Manton, geodesics in the moduli space determine the slow motion of topological defects , [1]. The adiabatic principle [2] has been successfully applied to black holes [3], magnetic monopoles [4], and, self-dual vortices both in Higgs, [5], and Chern-Simons-Higgs, [6]-[7]-[8]-[9], models. Recently, the moduli space of BPS domain walls has been discussed by Tong [10] and, following Manton's method, the low energy dynamics of solitons has been studied by Townsend and Portugues [11] in variations of the Wess-Zumino model.

In this paper we shall study the low energy dynamics of BPS kinks living in a topological sector of a super-symmetric (1+1)-dimensional system proposed by Bazeia and co-workers in [13]. Moreover, Shifman and Voloshin in [14] have shown that the (1+1)D system comes from the dimensional reduction of a generalized Wess-Zumino model with two chiral super-fields; in this latter case, the kink solutions appear as BPS domain walls.

From a one-dimensional perspective, the system encompasses several topological sectors and Shifman and Voloshin in [14] discovered that there exists a degenerate family of BPS domain walls in a distinguished sector. In [15], three of us found that for some critical values of the coupling parameter there exist more degenerate kink families in other topological sectors, although the new walls are generically non-BPS.

Every BPS domain wall, however, seems to be made from two basic walls, which belong to other topological sectors. This interesting structure has been explored by Sakai and Sugisaka, who found in [16] an intriguing bound-state of wall/anti-wall pairs. The aim of this work is to describe how the basic kinks move in the moduli space of BPS topological kinks. To achieve this goal we shall apply Manton's method and we shall thus extend the applicability of the adiabatic principle to one-dimensional topological defects.

The organization of the paper is as follows: in Section §2 we briefly summarize the general framework of (1+1)D super-symmetric field theory. Section §3 is devoted to describing the moduli space of BPS super-kink solutions. In Section §4 we unveil the metric inherited from the adiabatic kink motion in the moduli space, determine the geodesic orbits, and describe the low speed motion of BPS kinks. Finally, in Section §5 we interpret these results from the point of view of moving walls.

2 $\mathcal{N}=1$ super-symmetric (1+1)-dimensional field-theoretical systems

Using non-dimensional space-time coordinates and fields as defined in Reference [15], we consider Bose, $\hat{\vec{\phi}}(x^\mu)=\sum_{a=1}^2\hat{\phi}^a(x^\mu)\vec{e}_a$, and Fermi,

$$\hat{\vec{\psi}}(x^\mu)=\sum_{a=1}^2\left(\begin{array}{c}\hat{\psi}_1^a(x^\mu)\\ \hat{\psi}_2^a(x^\mu)\end{array}\right)\vec{e}_a \qquad ,$$

fields. Here, $x^\mu=(x^0,x^1)$ are local coordinates in $\mathbb{R}^{1,1}$ space-time, $\vec{e}_a.\vec{e}_b=\delta_{ab}$ is an ortho-normal basis in $\mathbb{R}^2$ internal (iso-spin) space, and the Fermi fields belong to the Majorana representation of the Spin$(1,1;\mathbb{R})$ group: if $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$, $g^{\mu\nu}=\mathrm{diag}(1,-1)$, is the Clifford algebra of $\mathbb{R}^{1,1}$, we choose $\gamma^0=\sigma^2,\gamma^1=i\sigma^1$. Also, $\gamma^5=\gamma^0\gamma^1=\sigma^3$ with $\sigma^1,\sigma^2,\sigma^3$ the 2×2 Pauli matrices.

The canonical quantization procedure dictates the equal-time commutation/anticommutation relations among the fields and their momenta. Defining $\hat{\vec{\pi}}(x^\mu)=\frac{\partial\hat{\vec{\phi}}}{\partial x^0}(x^\mu)$, we have that:

$$[\hat{\phi}^a(x),\hat{\pi}^b(y)]=i\delta^{ab}\delta(x-y) \qquad , \qquad \{\hat{\psi}^a_\alpha(x),\hat{\psi}^b_\beta(y)\}=\delta^{ab}\delta_{\alpha\beta}\delta(x-y) \quad , \tag{1}$$

where $\alpha,\beta=1,2$ are Majorana spinor indices and a natural system of units -$\hbar=c=1$- has been chosen.

Interacting $\mathcal{N}=1$ super-symmetric field theory is built from the normal-ordered super-charge operator:

$$\hat{Q}=\int dx: \left[\gamma^\mu\gamma^0\hat{\vec{\psi}}(x^\mu)\partial_\mu\hat{\vec{\psi}}(x^\mu)+i\gamma^0\hat{\vec{\psi}}(x^\mu)\vec{\nabla}\hat{W}(x^\mu)\right]: \qquad . \tag{2}$$

Interactions come from the gradient of the super-potential $\vec{\nabla}\hat{W}(x^\mu)=\sum_{a=1}^2\frac{\partial\hat{W}}{\partial\phi^a}(x^\mu)\vec{e}_a$ and the super-symmetry algebra:

$$\{\hat{Q}_\alpha,\hat{Q}_\beta\}=2(\gamma^\mu\gamma^0)_{\alpha\beta}\hat{P}_\mu-2i\gamma^1_{\alpha\beta}\hat{T} \quad , \quad \alpha,\beta=1,2 \tag{3}$$

encompasses the energy,

$$\hat{P}_0=\frac{1}{2}\int dx: \left[\hat{\vec{\pi}}\hat{\vec{\pi}}+\frac{\partial\hat{\vec{\phi}}}{\partial x}\frac{\partial\hat{\vec{\phi}}}{\partial x}+\vec{\nabla}\hat{W}\vec{\nabla}\hat{W}-i\hat{\bar{\vec{\psi}}}\gamma^1\frac{\partial\hat{\vec{\psi}}}{\partial x}+\hat{\bar{\vec{\psi}}}\vec{\Delta}\hat{W}\hat{\vec{\psi}}\right]: \qquad , \tag{4}$$

the momentum,

$$\hat{P}_1 = -\int dx : \left[\hat{\vec{\pi}} \frac{\partial \hat{\vec{\psi}}}{\partial x} + \frac{i}{2} \hat{\vec{\psi}}^t \frac{\partial \hat{\vec{\psi}}}{\partial x} \right] : \qquad , \tag{5}$$

and the anomalous topological/central charge,

$$\hat{T} = \int dx : \left[\frac{\partial \hat{\vec{\phi}}}{\partial x} . \vec{\nabla} \left(\hat{W} + \frac{1}{4\pi} \Delta \hat{W} \right) \right] : \quad , \tag{6}$$

operators. In formulas (4)-(5)-(6) we have defined $\hat{\bar{\vec{\psi}}}(x^\mu) = \hat{\vec{\psi}}^t(x^\mu)\gamma^0$, and

$$\vec{\vec{\Delta}} \hat{W} = \vec{\nabla} \otimes \vec{\nabla} \hat{W} = \sum_{a=1}^{2} \sum_{b=1}^{2} \vec{e}_a \otimes \vec{e}_b \frac{\partial^2 \hat{W}}{\partial \phi^a \partial \phi^b} \quad , \Delta \hat{W} = \vec{\nabla} \vec{\nabla} \hat{W} = \sum_{a=1}^{2} \frac{\partial^2 \hat{W}}{\partial \phi^a \partial \phi^a}$$

are respectively the Hessian and Laplacian operators applied to $\hat{W}$. Note that there is no anomaly in the central charge if the super-potential is harmonic, the condition for the existence of $\mathcal{N} = 2$ super-symmetry.

The usual definition of BPS states in this system, [12], is derived from (3):

$$\hat{P}_0 = \frac{1}{2} \left(\hat{Q}_1 \pm \hat{Q}_2 \right)^2 + |\hat{T}| \qquad , \qquad \left(\hat{Q}_1 \pm \hat{Q}_2 \right) |BPS\rangle = 0 \tag{7}$$

as the requirement of minimal energy in each super-selection sector. To find these states a variational method, using the coherent states

$$\hat{\vec{\pi}}(x)|\vec{\phi}(x), \vec{\psi}(x)\rangle = \vec{0} \ , \ \hat{\vec{\phi}}(x)|\vec{\phi}(x), \vec{\psi}(x)\rangle = \vec{\phi}(x)|\vec{\phi}(x), \vec{\psi}(x)\rangle \ , \ \hat{\vec{\psi}}(x)|\vec{\phi}(x), \vec{\psi}(x)\rangle = \vec{\psi}(x)|\vec{\phi}(x), \vec{\psi}(x)\rangle$$

as trial states, is conventionally applied. $\vec{\phi}(x)$ and $\vec{\psi}(x)$ are respectively scalar and Majorana spinor static classical field configurations and, on these states, the (7) BPS condition becomes:

$$\vec{\psi}_\mp(x) \left(\frac{d\vec{\phi}}{dx} \pm \vec{\nabla} W \right) |\vec{\phi}(x), \vec{\psi}(x)\rangle = 0 \qquad , \vec{\psi}_\pm(x) = \vec{\psi}_1(x) \pm \vec{\psi}_2(x) \tag{8}$$

because the expectation values in this kind of state of normal-ordered operator functionals are equal to their classical counterparts, see e.g. [17].

Thus, the BPS states are the coherent states for which the scalar field configurations correspond to the flow lines of $\mp \mathrm{grad} W$ whereas the corresponding γ^1 eigen-spinor $\vec{\psi}_\pm$ configurations are zero. Moreover, super-symmetry forces the surviving (non-null) spinor configurations to satisfy the equation:

$$\frac{d\vec{\psi}_\mp}{dx}(x) = -\vec{\vec{\Delta}} W(\vec{\phi}) \vec{\psi}_\mp(x) \tag{9}$$

3 The BPS kink moduli space

Our choice of super-potential is:

$$W(\vec{\phi}) = 4\sqrt{2}\left[\frac{1}{3}(\vec{\phi}\cdot\vec{e}_1)^3 - \frac{1}{4}\vec{\phi}\cdot\vec{e}_1(1-(\vec{\phi}\cdot\vec{e}_2)^2)\right] \qquad .$$

Thus, we deal with the super-symmetric extension of the BNRT model analyzed in [15] for the special value of the parameter $\sigma = \frac{1}{2}$.

In Reference [15]; the BPS kinks were shown to be in one-to-one correspondence with the kink orbits:

$$\phi_1^2 + \frac{1}{2}\phi_2^2 = \frac{1}{4} + c\phi_2^4 \qquad . \tag{10}$$

The explicit analytical expressions for these families of BPS kink/anti-kink solutions are:

$$\vec{\phi}^K[x;a,b] = \pm\left(\frac{1}{2}\frac{\sinh\left(2\sqrt{2}(x+a)\right)}{\cosh\left(2\sqrt{2}(x+a)\right)+b^2}\vec{e}_1 + \frac{b}{\sqrt{b^2+\cosh\left(2\sqrt{2}(x+a)\right)}}\vec{e}_2\right) \tag{11}$$

for arbitrary integration constants $a, b \in (-\infty,\infty)$. The constant b is defined in terms of $c \in (-\infty, \frac{1}{4})$ as: $b^2 = \frac{1}{\sqrt{1-4c}}$, see [15]. The ϕ_1- and ϕ_2-components of these solitary waves are shown in Figure 1 for some particular values of b.

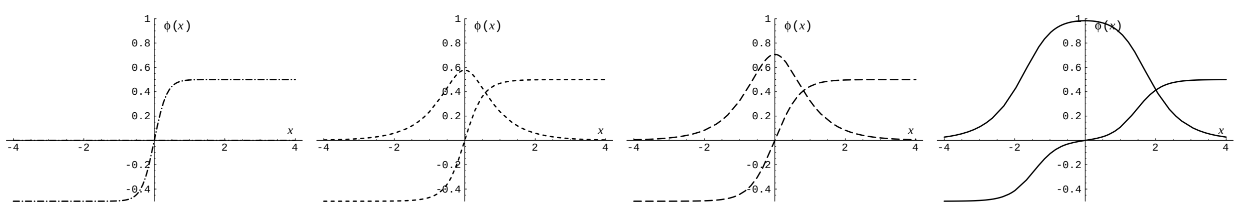

Figure 1: Solitary waves (11) corresponding to: *(a)* $b=0$, *(b)* $b=\sqrt{0.5}$, *(c)* $b=1$ and *(d)* $b=\sqrt{30}$.

The associated BPS fermionic form factors, the solutions of (9) for $\vec{\phi}^K$, are:

$$\vec{\psi}^K_{\pm}(x) = 0 \qquad , \quad \vec{\psi}^K_{\mp}[x;d,f] = \left[d\frac{\partial\vec{\phi}^K}{\partial a}(x) + f\frac{\partial\vec{\phi}^K}{\partial b}(x)\right]\varepsilon_{\mp} \quad , \tag{12}$$

where d, f are real integration constants and $\varepsilon_{\mp} = \varepsilon_1^{K\mp} \mp \varepsilon_2^{K\mp} = 2$; $\varepsilon^{K\mp} = \begin{pmatrix} 1 \\ \mp 1\end{pmatrix}$ are constant eigen-spinors of γ^1. (See Figures 2 and 3, where the fermionic partners of the bosonic solitary waves for the same values of the b parameter as above, are plotted) .

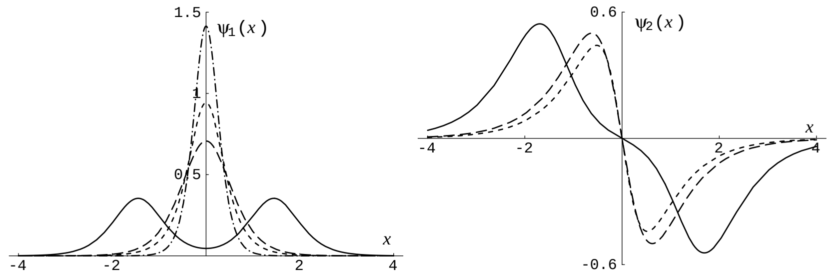

Figure 2: First *(a)* and second *(b)* components of $\frac{\partial \vec{\phi}^K}{\partial a}$.

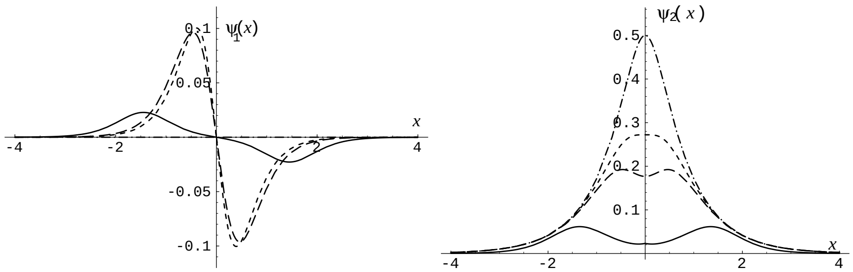

Figure 3: First *(a)* and second *(b)* components of $\frac{\partial \vec{\phi}^K}{\partial b}$.

Taking quotient by $\phi_2 \to -\phi_2$ both in (10) and (11) we find the moduli space $\mathcal{M}^K$ of BPS kinks. Thus, a and b^2 are good coordinates in $\mathcal{M}^K$, which is isomorphic to the upper half-plane: $\mathbb{H} = (-\infty, \infty) \times [0, \infty)$. Note, however, that the $\phi_2 \to -\phi_2$ reflection in the space of BPS kink solutions is tantamount to replacing $+\sqrt{b^2}$ by $-\sqrt{b^2}$ in (11), i.e. change b from positive to negative. b, therefore, is a good coordinate in the space of solutions.

The properties of a solitary wave determined by a point in $\mathcal{M}^K$ are encoded in the bosonic energy density:

$$\mathcal{E}^K[x;a,b] = \frac{\partial \vec{\phi}^K}{\partial x} . \frac{\partial \vec{\phi}^K}{\partial x} = \frac{4 + 7b^2 \cosh[2\sqrt{2}(x+a)] + 4b^4 \cosh[4\sqrt{2}(x+a)] + b^2 \cosh[6\sqrt{2}(x+a)]}{2(b^2 + \cosh[2\sqrt{2}(x+a)])^4} \tag{13}$$

Figure 4 shows a plot of $\mathcal{E}^K[x;a,b]$ for $a = 0$ and the same values of b as in the above Figures. The physical meaning of $-a$ is obvious: it is the center of mass of the solitary wave. There are two regimes in the b^2-parameter classified by the dependence on b^2 of the critical points of the energy density, i.e. the zeroes of:

$$\begin{aligned} \frac{\partial \mathcal{E}^{\mathcal{K}}}{\partial x}[x;a,b] &= \frac{4\sqrt{2}\sinh 2\sqrt{2}(x+a)}{(b^2 + \cosh 2\sqrt{2}(x+a))^5} \, P_3(\cosh 2\sqrt{2}(x+a)) \qquad ; \\ P_3(\cosh 2\sqrt{2}(x+a)) &= -b^2 \cosh^3 2\sqrt{2}(x+a) - b^4 \cosh^2 2\sqrt{2}(x+a) + \\ &+ (-3b^2 + 4b^6)\cosh 2\sqrt{2}(x+a) + 5b^4 - 4 \qquad . \end{aligned}$$

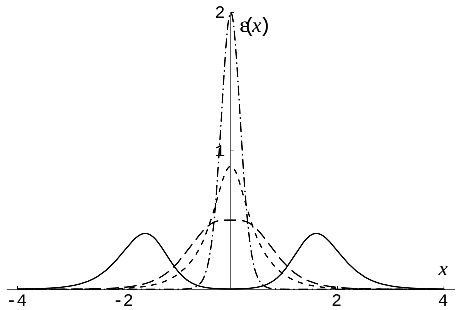

Figure 4: Energy density $\mathcal{E}^K[x;0,b]$.

Apart from the obvious solution, $x=-a$, i.e. $\sinh 2\sqrt{2}(x+a)=0$, $\forall b^2$, we can classify the solutions of $\frac{\partial \mathcal{E}^K}{\partial x}[x;a,b]=0$ in terms of the roots of the cubic polynomial $P_3(\cosh 2\sqrt{2}(x+a))$. Writing P_3 as $P_3(\cosh 2\sqrt{2}(x+a))=-b^2\tilde{P}(u)$, where $\tilde{P}(u)$ is the bi-cubic polynomial $\tilde{P}(u)=(u^2)^3+(b^2+3)(u^2)^2-(4b^4-2b^2-6)(u^2)-4\left(b^4+b^2-1-\frac{1}{b^2}\right)$ in the variable $u^2(x)=-1+\cosh 2\sqrt{2}(x+a)$, a classical analysis of the roots of $\tilde{P}(u)$, based on the Cardano and the Vieta formulae and use of the Rolle's theorem, shows that:

- $\tilde{P}(u)$ has no real roots in $u(x)$ if $b^2\in[0,1]$. Thus, $\frac{\partial \mathcal{E}^K}{\partial x}[x;a,b]=0$ only for $x=-a$, which is the only critical point of $\mathcal{E}^K$ as a function of x. Moreover, $\mathcal{E}^K[-a;a,b]=\frac{2}{(b^2+1)^2}$ is the maximum value of $\mathcal{E}^K$ on the real line; b^2, therefore, measures the height of the solitary wave energy density , see Figure 4.

- Things are more interesting if $b^2\in(1,\infty)$: $\tilde{P}$ has two real roots. As a cubic polynomial in u^2, $\tilde{P}$ has a single positive root $r(b^2)$ that depends on the value of b^2; hence, $u=\pm\sqrt{r}$ are the real roots of $\tilde{P}(u)$. Besides $x=-a$, which is a minimum, two other critical points of $\mathcal{E}^K$ arise at $x=-a\pm m(b^2)$ if $b^2>1$, where $m(b^2)=\frac{1}{2\sqrt{2}}\text{arccosh}(1+r(b^2))$.

  These two points are maxima of $\mathcal{E}^K$ and the solitary wave is made from two lumps if $b^2>1$. The distance between the peaks grows with b^2 and b^2 must be understood as the relative coordinate of a system of two "particles" if $b^2>1$, whereas $-a$ is still the center of mass coordinate.

  Alternatively, one can trust Mathematica and just look at Figure 4.

In fact, exactly at the $b^2=\infty$ limit, the solutions

$$\vec{\phi}^B[x;a]=\frac{(-1)^\gamma}{4}(1-(-1)^\alpha\tanh(\sqrt{2}(x+a))\vec{e}_1+(-1)^\beta\sqrt{\frac{1}{2}(1+(-1)^\alpha\tanh(\sqrt{2}(x+a))}\vec{e}_2\ , \quad (14)$$

$(\alpha,\beta,\gamma=0,1)$ living in other topological sectors, appear, see Figure 5. We can think of these kinks as the basic BPS solitary waves of the system, although they are not accessible in the kink

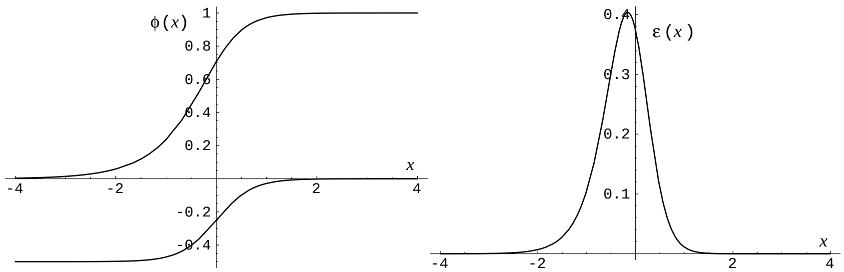

Figure 5: Solitary waves (14) *(a)* and its energy density *(b)*.

moduli space discussed above. $\vec{\phi}^B[x;a]$ are also flow lines of grad$W$ but start or end at a saddle point $\pm\vec{e}_2$ of W and only reach the minimum $\frac{1}{2}\vec{e}_1$ of W in the first case.

There are two other critical values of b^2 worthy of note. First, when $b^2=1$ the two basic kinks merge and become fused in a single kink. We shall denote this special kind of kinks as TK2E, because they are topological kinks with two non-null components restricted to living on the ellipse given by (10) for $c=0$. Second, the TK1 kinks with only one non-null component are reached at $b^2=0$.

4 Geodesic motion in the kink moduli space

In our framework, the adiabatic principle is equivalent to restricting time-evolution to the subspace of BPS kink states. The expectation value of the kinetic energy in these states is:

$$K = \frac{1}{2}\int \quad dx \quad \langle\vec{\phi}^K[x;a(t),b(t)]|\!:\hat{\vec{\pi}}(x)\hat{\vec{\pi}}(x)\!:|\vec{\phi}^K[x;a(t),b(t)]\rangle = $$
$$= \quad \frac{1}{2}g_{aa}(a,b)\dot{a}^2 + g_{ab}(a,b)\dot{a}\dot{b} + \frac{1}{2}g_{bb}(a,b)\dot{b}^2 \qquad (15)$$

where:

$$g_{aa}(a,b) = \int_{-\infty}^{\infty} dx \frac{\partial\vec{\phi}^K}{\partial a}\cdot\frac{\partial\vec{\phi}^K}{\partial a}; \quad g_{ab}(a,b) = \int_{-\infty}^{\infty} dx \frac{\partial\vec{\phi}^K}{\partial a}\cdot\frac{\partial\vec{\phi}^K}{\partial b}; \quad g_{bb}(a,b) = \int_{-\infty}^{\infty} dx \frac{\partial\vec{\phi}^K}{\partial b}\cdot\frac{\partial\vec{\phi}^K}{\partial b} \qquad (16)$$

We think of K as the Lagrangian for geodesic motion in the moduli space with the metric tensor:

$$g_{aa}(a,b) = \frac{2\sqrt{2}}{3} \quad ; \quad g_{ab}(a,b) \ = \ 0 \quad ; \quad g_{bb}(a,b) = \frac{2\sqrt{2}}{3}f(b)$$
$$f(b) \ = \ \frac{1}{4(b^4-1)^2}\left[2b^6 - 5b^2 + 3\frac{\arctan\left(\frac{\sqrt{1-b^4}}{b^2}\right)}{\sqrt{1-b^4}}\right] \qquad (17)$$

Computation of the integrals (16) has been performed by changing variables to $y=\exp[2\sqrt{2}(x+a)]$ in such a way that the quadratures reduce to rational definite integrals in y. A subtle point is the following: we are using the b-parameter in formulas (16)-(17) . Because the good coordinate to describe the kink moduli space is b^2, we should understand the metric as a function of b^2. Nevertheless, it is sometimes useful to extend the reasoning concerning the geodesic behaviour to negative values of b in order to describe slow motion in the space of kink solutions.

The transition from one to two lumps is seen in formula (17) in the trading of arctan by $i\,$arctanh that happens at $b = 1$. $f(b)$, however, is also real for $b^2 > 1$ because the denominator becomes purely imaginary in this regime, see Figure 6.

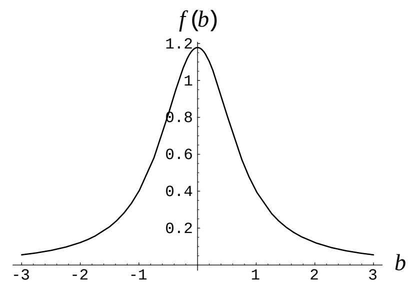

Figure 6: Graphic of the function $f(b)$.

The geodesics of a metric in the space of kink solutions of the form :

$$ds^2 = \frac{2\sqrt{2}}{3}\left(da^2 + f(b)\, db^2\right)$$

are easily found (at least implicitly). Writing the metric in terms of a new variable $c = \int \sqrt{f(b)}db$, we have :

$$\frac{3}{2\sqrt{2}}\, ds^2 = da^2 + dc^2 \quad , \quad dc = \sqrt{f(b)}\, db$$

and the geodesic curves are straight lines in the $a - c$ plane. If k_1, k_2, k_1', k_2', are integration constants, the geodesics are:

$$a(t) = k_1\, t + k_2 \qquad , \qquad c(t) = \int \sqrt{f(b)}\, db = k_1'\, t + k_2' \quad . \tag{18}$$

In terms of the new integration constants $\kappa_1 = \frac{k_1'}{k_1}$, $\kappa_2 = k_2' - \kappa_1 k_2$, we can also represent the geodesic paths by writing c as a function of a:

$$c = \int \sqrt{f(b)}db = \kappa_1 a + \kappa_2 \quad . \tag{19}$$

We immediately identify a simple kind of geodesic motion: taking $\kappa_1 = 0$ in (19), orbits with $b =$ constant are found. This motion corresponds to free displacement of the kink center of mass, for all types of solitary wave, without changing the shape of the kink profile, see Figure 7.

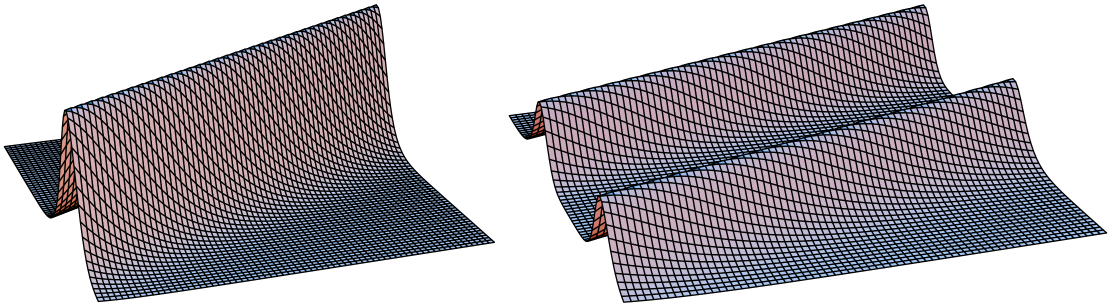

Figure 7: Energy density evolution along straight geodesic lines with b=constant: *(a)* $b = 0.9$, a single lump is moving *(b)* $b = 10$, synchronous motion of two lumps. Time runs from left to right

To search for the more general geodesic motion, the problem is to find the explicit form of $\int \sqrt{f(b)}db$. Because explicitly performing this integration is out of reach by analytical means, we propose two alternative ways of describing the geodesic motion of BPS kinks.

4.1 Numerical integration

In Figure 8(a) a Mathematica numerical plot of several geodesic orbits (19) is shown. κ_1 has been set to 3, 2 and 1 whereas the freedom in κ_2 has been fixed by setting the value of $b^2 = 0.01$ -near the TK1 point in the moduli space- at the instant $t = -\frac{k_2}{k_1}$. The features common to these generic geodesics are as follows: the starting point is a point in the moduli space close to the $\vec{\phi}^B[x;a]$ kinks (14). Coming from very far apart, the two basic lumps begin to approach each other when b^2 decreases. This approach occurs simultaneously to a global displacement of the center of mass: a increases. The two kinks merge in a single lump at the point $b^2 = 1$ in the moduli space, and then, move together getting higher and thinner until becoming the TK1 kink when $b = 0$. At this point a spectacular rebound takes place and the process repeats itself backwards. The composite lump moves, becoming shorter and thicker until the critical value $b^2 = 1$ is reached again, where a meiosis takes place, giving back rise to two separate lumps. The geodesic evolution is completed through the increasing separation of the two lumps, b^2 increasing , and the center of mass motion, a increasing , asymptotically running back towards the two basic $\vec{\phi}^B[x;a]$ kinks (14).

The closer look at the geodesic orbits near the rebound point depicted in Figure 9 shows us that the bigger κ_1 the shorter the time that the two lumps remain aggregated. At the $\kappa_1 = \infty$ limit, the geodesic curve becomes a vertical straight line in the moduli space and there is no motion of the center of mass at all. Different values of κ_2 give different geodesic orbits by setting the kink point crossed at the instant $t = -\frac{k_2}{k_1}$.

Finally, in Figure 8(b) we have plotted the same geodesics, now in the space of kink solutions, not in the moduli space, by allowing negative values of b. The graphics reveal that the rebound is in

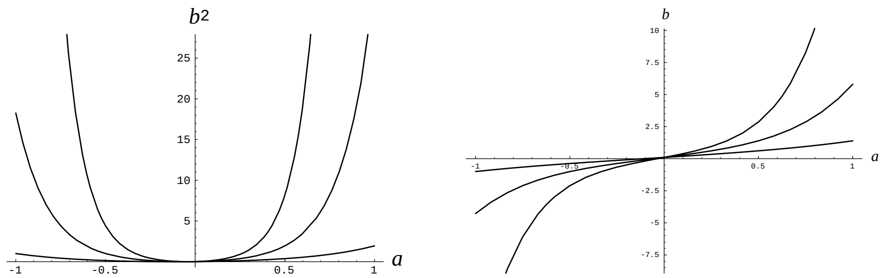

Figure 8: Geodesic orbits. In the $a - b^2$ half-plane *(a)*. In the $a - b$ plane *(b)*.

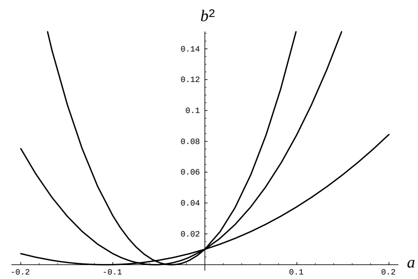

Figure 9: Detail of the geodesic orbits of Figure 8*(a)* near the TK1 point.

fact smooth: the transition through the TK1 point is accompanied by a flip of the ϕ_2^K-component of the kink profile to $-\phi_2^K$ in such a way that the process ends in a different set of two basic kinks. The motion starts from two $\vec{\phi}^B[x;a]$ kinks (14) living in the topological sectors in which any configuration asymptotically connects the vacuum $\vec{v}_-^A = -\frac{1}{2}\vec{e}_1$ with the vacuum $\vec{v}_+^B = \vec{e}_2$, and, $\vec{v}_+^B$ with $\vec{v}_+^A = \frac{1}{2}\vec{e}_1$. The end point, however, is very close to two $\vec{\phi}^B[x;a]$ kinks (14) connecting $\vec{v}_-^A = -\frac{1}{2}\vec{e}_1$ with $\vec{v}_-^B = -\vec{e}_2$, and, $\vec{v}_-^B$ with $\vec{v}_+^A = \frac{1}{2}\vec{e}_1$.

The whole picture is synthesized in Figure 10. The energy density along the geodesic $\kappa_1 = 2$, $k_1 = 1$, $k_2 = 0$ is plotted as a function of x and t, showing the adiabatic evolution of the two basic kinks. In the drawing, time runs from left to right and the spatial coordinate x grows from bottom to top.

4.2 Asymptotic behaviour of geodesics

It is possible and indeed appropriate to find analytically the geodesics near the boundary of the moduli space, the union of the $b = 0$ axis and the half-circle at infinity, $b = +\infty$.

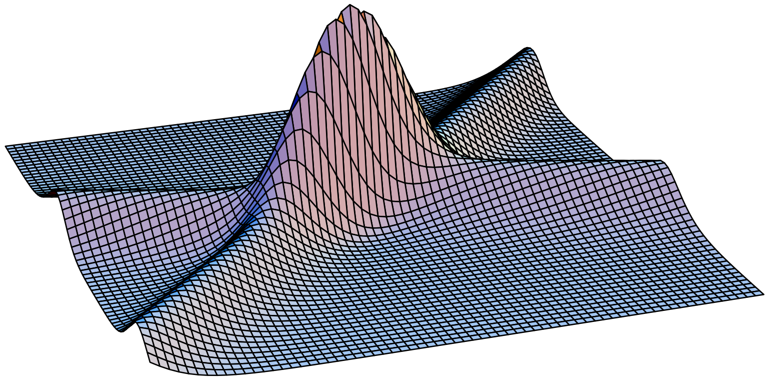

Figure 10: Evolution of energy density evolution along a generic geodesic curve.

4.2.1 $b \approx 0$

Close to the TK1 point in the moduli space, the metric can be approximately obtained from the series expansion of $f(b)$ around $b = 0$:

$$f(b) = \frac{3\pi}{8} - 2b^2 + \mathcal{O}(b^4) \qquad .$$

The geodesic orbits are given in this region by :

$$\kappa_1 a + \kappa_2 = \int \sqrt{f(b)} db \simeq \int \sqrt{\frac{3\pi}{8} - 2b^2}\, db = \frac{4\, b\, \sqrt{3\,\pi - 16\, b^2} + 3\,\pi\, \arcsin(\frac{4\, b}{\sqrt{3\,\pi}})}{16\,\sqrt{2}} \qquad . \tag{20}$$

Therefore, equation (20) analytically describes how the TK1 kink is reached from kink configurations in its neighbourhood and vice-versa. Note that this statement is tantamount to saying that (20) determines the rebound of the two aggregated kinks on the TK1 kink.

4.2.2 $b^2 \approx \infty$, $b \approx \pm\infty$

The expansion of $f(b)$ around $b = +\infty$;

$$f(b) = \frac{1}{2}\frac{1}{b^2} + \mathcal{O}(\frac{1}{b^6}) \qquad ,$$

leads to the geodesic asymptotic behaviour

$$\kappa_1 a + \kappa_2 = \int \sqrt{f(b)}\, db \simeq \frac{1}{\sqrt{2}} \ln |b| \qquad ,$$

which shows how the two $\vec{\phi}^B[x; a]$ kinks (14) are reached exponentially fast in a, or, vice-versa, how fast the two lumps start to approach each other.

4.2.3 $b^2 \approx 1$

There is still a third special point corresponding to a two-component topological kink where the melting into a single lump takes place or the splitting into two lumps. This TK2E kink lives on a half-ellipse given by (10) when $c = 0$. Despite appearances, the metric at $b = 1$ is regular and the series expansion of $f(b)$ in the vicinity of this point reads:

$$f(b) = \frac{2}{5} - \frac{4}{7}(b-1) + \mathcal{O}((b-1)^2) \qquad .$$

The geodesic equations

$$\kappa_1 a + \kappa_2 = \int \sqrt{f(b)}db \simeq \frac{-7}{6}\left(\frac{34}{35} - \frac{4}{7}b\right)^{\frac{3}{2}}$$

analytically rule, therefore, the low energy process of two-lump fusion into TK2E / TK2E fission into two lumps in the vicinity of the TK2E kink point of the moduli space .

5 From low energy kink dynamics to slow motion of domain walls

We finish this paper by offering some comments about the importance of the model that we have studied within the context of the low energy effective theories inspired in string/M theory. In Reference [14], the authors analyzed a generalized $\mathcal{N} = 1$ super-symmetric Wess-Zumino model in $(3+1)$-dimensional space-time with two chiral super-fields , Φ_1, Φ_2 , and interactions determined by the super-potential:

$$W(\Phi_1, \Phi_2) = \frac{4}{3}\Phi_1^3 - \Phi_1 + 2\sigma\Phi_1\Phi_2^2 \qquad . \tag{21}$$

We implicitly assume non-dimensional field variables and that σ is a non-dimensional coupling constant between the two chiral super-fields. Distinguishing among real and imaginary parts for the super-fields, $\Phi_1 = \phi_1 + i\psi_1$, $\Phi_2 = \phi_2 + i\psi_2$, the real and imaginary parts of the super-potential $W = W^1 + iW^2$ read:

$$\begin{aligned} W^1(\Phi_1, \Phi_2) &= \phi_1[\frac{4}{3}\phi_1^2 - 4\psi_1^2 + 2\sigma(\phi_2^2 - \psi_2^2) - 1] - 4\sigma\psi_1\phi_2\psi_2 \\ W^2(\Phi_1, \Phi_2) &= \psi_1[4\phi_1^2 - \frac{4}{3}\psi_1^2 + 2\sigma(\phi_2^2 - \psi_2^2) - 1] + 4\sigma\phi_1\phi_2\psi_2 \quad . \end{aligned} \tag{22}$$

Therefore, the restriction to the real part of the model -$\psi_1 = \psi_2 = 0, W^2 = 0$ - leads to the BNRT system proposed in [13] and discussed from the point of view of kink defects in [15]. In fact, the BPS kinks described in [15] are in one-to-one correspondence with the BPS domain walls discovered in [14]. In particular, the moduli space of BPS kinks is identical to the moduli space of BPS walls of the generalized Wess-Zumino model and all that we have concluded for the slow motion of BPS kinks in the $\sigma = \frac{1}{2}$ case can safely be translated to the adiabatic motion of BPS walls in the corresponding generalized Wess-Zumino model.

This combination of dimensional reduction and reality conditions poses a problem from a (1+1)-dimensional perspective. We denote by $\mathcal{W} = W^1$ the real part of the reduced super-potential. It has been shown in Reference [14] that there is a partner "quasi-super-potential" $\tilde{\mathcal{W}}$ in the sense that the generalized Cauchy-Riemann equations

$$\frac{\partial \mathcal{W}}{\partial \phi_a} = 2\sigma \varepsilon_{ab} |\phi_2|^{\frac{2}{\sigma}+1} \frac{\partial \tilde{\mathcal{W}}}{\partial \phi_b} \qquad ; \varepsilon_{ab} = -\varepsilon_{ba} \quad , a = 1, 2 \quad , b = 1, 2 \qquad . \tag{23}$$

are satisfied . $\mu(\phi_2) = 2\sigma|\phi_2|^{\frac{2}{\sigma}+1}$ is the integrating factor used in Reference [15] to find the flow lines of grad$\mathcal{W}$. In fact, the solutions of the first-order equations (9) in [15] - written here in (10) for $\sigma = \frac{1}{2}$ - satisfy:

$$\frac{\partial \mathcal{W}}{\partial \phi_2} d\phi_1 - \frac{\partial \mathcal{W}}{\partial \phi_1} d\phi_2 = \mu(\phi_2) d\tilde{\mathcal{W}} = 0 \quad .$$

Thus, $d\tilde{\mathcal{W}}$ is an exact one-form on the solutions (in $\mathbb{R}^2$) and $\tilde{\mathcal{W}}$ remains constant on the kink orbits. Only if $\sigma = -2$ will the dimensional reduction that we are considering coincide with the outcome of the standard dimensional reduction in the genuine Wess-Zumino model. In this last case, there is no need for any integrating factor because (23) becomes strictly the Cauchy-Riemann equations and $\mathcal{W}, \tilde{\mathcal{W}}$ are conjugate harmonic functions that allow a complex super-potential to be built . If $\sigma \neq -2$ this is not so and there is no possibility of obtaining $\mathcal{N} = 2$ (1+1)-dimensional super-symmetry, which, in turn, means that one must expect one-loop corrections in the surface tension of the walls.

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